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Existence results for a class of nonlocal problems involving $p(x)$-Laplacian

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Abstract We study the existence of weak solutions for a $p(x)$-Kirchhoff problem. The main tool used is the variational method, more precisely, the Mountain Pass Theorem.

Mathematics Subject Classification 35J48 · 35J66

الملخص

ندرس وجود حلول ضعيفة لمسألة كيرشوف-$p(x)$. الأداة الرئيسية التي تم استخدامها هي طريقة التغايراتية، وبشكل أدق مبرهنة الممر الجبلي.

1 Introduction

In this paper, we consider the following problem:

$$\begin{aligned} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) &= \lambda(a(x)|u|^{q(x)-2}u+b(x)|u|^{r(x)-2}u) \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary $\partial\Omega$, $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, λ is a positive number, p is Lipschitz continuous on $\overline{\Omega}$, and q, r are continuous functions on $\overline{\Omega}$ with $q^- := \inf_{x\in\overline{\Omega}} q(x) > 1$, $r^- := \inf_{x\in\overline{\Omega}} r(x) > 1$, $a(x), b(x) > 0$ for $x \in \overline{\Omega}$ such that $a \in L^{\alpha(x)}(\Omega)$, $\alpha(x) = \frac{p(x)}{p(x)-q(x)}$ and $b \in L^{\gamma(x)}(\Omega)$, $\gamma(x) = \frac{p^*(x)}{p^*(x)-r(x)}$. Hereafter,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N;\\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

We will use the notations such as h^- and h^+, where

$$h^- := \inf_{x\in\overline{\Omega}} h(x) \leq h(x) \leq h^+ := \sup_{x\in\overline{\Omega}} h(x) < +\infty.$$

The operator $\triangle_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated properties than the p-Laplacian; for

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example, it is inhomogeneous. The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics [21] and image restoration [7]. The problem (1.1) is a generalization of a model introduced by Kirchhoff [18]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. Lions [19] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions [19], various equations of Kirchhoff-type have been studied extensively, see [2,5]. The study of Kirchhoff-type equations has already been extended to the case involving the p-Laplacian (for details, see [3,4,10,11]) and $p(x)$-Laplacian (see [9,12]). Motivated by the above papers and the results in [8,20], we consider (1.1) to study the existence of weak solutions.

2 Preliminary

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For more details, see [13–15]. Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) \quad \text{and} \quad \inf_{x\in\overline{\Omega}} p(x) > 1 \right\}.$$

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \quad \text{is measurable and} \quad \int_\Omega |u|^{p(x)}\,\mathrm{d}x < +\infty \right\},$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \tau > 0; \int_\Omega \left|\frac{u}{\tau}\right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = \inf \left\{ \tau > 0; \int_\Omega \left(\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \left|\frac{u}{\tau}\right|^{p(x)} \right) \mathrm{d}x \le 1 \right\}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Of course the norm $\|u\| = |\nabla u|_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. In this paper, we denote by $X = W_0^{1,p(x)}(\Omega)$.

Lemma 2.1 ([15]) *Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable and uniformly convex Banach spaces.*

Lemma 2.2 ([15]) *Hölder inequality holds, namely*
$\int_\Omega |uv|\,\mathrm{d}x \le 2|u|_{p(x)}|v|_{p'(x)} \quad \forall u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega)$, *where* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Lemma 2.3 ([6]) *Assume that $h \in L_+^\infty(\Omega)$, $p \in C_+(\overline{\Omega})$. If $|u|^{h(x)} \in L^{p(x)}(\Omega)$. Then we have*

$$\min\{|u|_{h(x)p(x)}^{h^-}, |u|_{h(x)p(x)}^{h^+}\} \le ||u|^{h(x)}|_{p(x)} \le \max\{|u|_{h(x)p(x)}^{h^-}, |u|_{h(x)p(x)}^{h^+}\}.$$

Lemma 2.4 ([14]) *Assume that Ω is bounded and smooth.*

- *Let p be Lipschitz continuous and $p^+ < N$. Then for $h \in L^{\infty}_{+}(\Omega)$ with $p(x) \leq h(x) \leq p^*(x)$ there is a continuous embedding $X \hookrightarrow L^{h(x)}(\Omega)$.*
- *Let $p \in C(\overline{\Omega})$ and $1 \leq q(x) < p^*(x)$ for $x \in \overline{\Omega}$. Then there is a compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$.*

Lemma 2.5 ([16]) *Set $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)}\,\mathrm{d}x$. Then for $u \in X$, we have*

(1) $\|u\| < 1$ *(respectively* $= 1; > 1$*) if and only if* $\rho(u) < 1$ *(respectively* $= 1; > 1$*);*
(2) *if* $\|u\| > 1$, *then* $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;
(3) *if* $\|u\| < 1$, *then* $\|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$.

Definition 2.6 A function $u \in X$ is said to be a weak solution of (1.1) if

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,\mathrm{d}x - \lambda\int_\Omega a(x)|u|^{q(x)-2}uv\,\mathrm{d}x$$
$$- \lambda\int_\Omega b(x)|u|^{r(x)-2}uv\,\mathrm{d}x = 0,$$

for all $v \in X$.

The Euler–Lagrange functional associated to (1.1) is

$$J_\lambda(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\right) - \lambda\int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)}\,\mathrm{d}x - \lambda\int_\Omega \frac{b(x)}{r(x)}|u|^{r(x)}\,\mathrm{d}x,$$

where $\widehat{M}(t) = \int_0^t M(s)\,\mathrm{d}s$. Then

$$\langle J'_\lambda(u), v\rangle = M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,\mathrm{d}x - \lambda\int_\Omega a(x)|u|^{q(x)-2}uv\,\mathrm{d}x$$
$$- \lambda\int_\Omega b(x)|u|^{r(x)-2}uv\,\mathrm{d}x,$$

for all $u, v \in X$, then we know that the weak solution of (1.1) corresponds to the critical point of the functional J_λ. Hereafter, $M(t)$ is supposed to verify the following assumptions:

(M_0) There exists $m_1 \geq m_0 > 0$ and $\mu \geq \nu > 1$ such that

$$m_0 t^{\nu-1} \leq M(t) \leq m_1 t^{\mu-1}.$$

(M_1) $\exists\, 0 < d < 1$ such that

$$\widehat{M}(t) \geq (1-d)M(t)t \quad \text{for all } t \geq 0.$$

An example of functions satisfying the assumptions (M_0) and (M_1):

$$M(t) = t\,\mathrm{Arctan}(t).$$

Throughout this paper, we assume the condition:

$$1 < q^- \leq q^+ < \nu p^-, \quad \max\left\{\mu p^+, \frac{p^+}{1-d}\right\} < r^- \leq r^+ < p^*(x) \quad \text{and} \quad p^+ < N. \tag{2.1}$$

For simplicity, we use C_i, $i = 1, 2 \ldots$, to denote the general positive constants whose exact values may change from line to line.

3 Main result

Theorem 3.1 *Assume p is Lipschitz continuous, $q, r \in C_+(\overline{\Omega})$ and Condition (2.1) is fulfilled. Then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, Problem (1.1) possesses a nontrivial weak solution.*

Lemma 3.2 *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $\rho, \tau > 0$ such that $J_\lambda(u) \geq \tau > 0$ for any $u \in X$ with $\|u\| = \rho$.*

Proof In view of Lemma 2.4, there exists a positive constant C_1 such that

$$|u|_{p(x)} \leq C_1\|u\|, \quad |u|_{p^*(x)} \leq C_1\|u\|, \quad for\ all\ \ u \in X. \tag{3.1}$$

Fix $\rho \in]0, 1[$ such that $\rho < \frac{1}{C_1}$. Then relation (3.1) implies $|u|_{p(x)} < 1$, $|u|_{p^*(x)} < 1$, for all $u \in X$ with $\|u\| = \rho$. By Lemmas 2.2 and 2.3, we obtain

$$\int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x \leq 2|a|_{\alpha(x)}||u|^{q(x)}|_{\frac{p(x)}{q(x)}} \leq 2|a|_{\alpha(x)}|u|^{q^-}_{p(x)}, \tag{3.2}$$

and

$$\int_\Omega b(x)|u|^{r(x)}\,\mathrm{d}x \leq 2|b|_{\gamma(x)}||u|^{r(x)}|_{\frac{p^*(x)}{r(x)}} \leq 2|b|_{\gamma(x)}|u|^{r^-}_{p^*(x)}, \tag{3.3}$$

for all $u \in X$. Combining (3.1), (3.2) and (3.3), we obtain

$$\int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x \leq 2|a|_{\alpha(x)}C_1^{q^-}\|u\|^{q^-} \quad \text{and} \quad \int_\Omega b(x)|u|^{r(x)}\,\mathrm{d}x \leq 2|b|_{\gamma(x)}C_1^{r^-}\|u\|^{r^-}, \tag{3.4}$$

for all $u \in X$. Hence, from (3.4) and (M_0) we deduce that for any $u \in X$ with $\|u\| = \rho$, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{m_0}{\nu}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\right)^\nu - \frac{\lambda}{q^-}\int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x - \frac{\lambda}{r^-}\int_\Omega b(x)|u|^{r(x)}\,\mathrm{d}x \\ &\geq \frac{m_0}{\nu(p^+)^\nu}\|u\|^{\nu p^+} - \frac{\lambda}{q^-}2|a|_{\alpha(x)}C_1^{q^-}\|u\|^{q^-} - \frac{\lambda}{r^-}2|b|_{\gamma(x)}C_1^{r^-}\|u\|^{r^-}. \end{aligned}$$

Putting

$$\lambda^* = \min\left\{\frac{m_0 q^- \rho^{\nu p^+ - q^-}}{4C_1^{q^-}\nu(p^+)^\nu|a|_{\alpha(x)}}, \frac{m_0 r^- \rho^{\nu p^+ - r^-}}{4C_1^{r^-}\nu(p^+)^\nu|b|_{\gamma(x)}}\right\}, \tag{3.5}$$

for any $u \in X$ with $\|u\| = \rho$, there exists $\tau = \frac{m_0\rho^{\nu p^+}}{2\nu(p^+)^\nu}$ such that $J_\lambda(u) \geq \tau > 0$ for any $\lambda \in (0, \lambda^*)$. This completes the proof. □

Lemma 3.3 *There exists $e \in X$ with $\|e\| > \rho$ (where ρ is given in Lemma 3.2) such that $J_\lambda(e) < 0$.*

Proof Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \neq 0$ and $t > 1$. By (M_0) we have

$$\begin{aligned} J_\lambda(t\varphi) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\varphi|^{p(x)}\,\mathrm{d}x\right) - \lambda\int_\Omega \frac{a(x)}{q(x)}|t\varphi|^{q(x)}\,\mathrm{d}x - \lambda\int_\Omega \frac{b(x)}{r(x)}|t\varphi|^{r(x)}\,\mathrm{d}x \\ &\leq \frac{m_1}{\mu}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\varphi|^{p(x)}\,\mathrm{d}x\right)^\mu - \lambda\frac{t^{q^-}}{q^+}\int_\Omega a(x)|\varphi|^{q(x)}\,\mathrm{d}x - \lambda\frac{t^{r^-}}{r^+}\int_\Omega b(x)|\varphi|^{r(x)}\,\mathrm{d}x \\ &\leq \frac{m_1}{\mu(p^-)^\beta}t^{\mu p^+}\left(\int_\Omega |\nabla\varphi|^{p(x)}\,\mathrm{d}x\right)^\mu - \lambda\frac{t^{q^-}}{q^+}\int_\Omega a(x)|\varphi|^{q(x)}\,\mathrm{d}x - \lambda\frac{t^{r^-}}{r^+}\int_\Omega b(x)|\varphi|^{r(x)}\,\mathrm{d}x. \end{aligned}$$

Since $\mu p^+ < r^-$, we obtain $\lim_{t\to\infty} J_\lambda(t\varphi) = -\infty$. Then for $t > 1$ large enough, we can take $e = t\varphi$ such that $\|e\| > \rho$ and $J_\lambda(e) < 0$. □

Lemma 3.4 *The functional J_λ satisfies the Palais–Smale condition (PS).*

Proof Suppose that $(u_n) \subset X$ is a (PS) sequence; that is,

$$\sup |J_\lambda(u_n)| \le C_2, \quad J'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

We prove that (u_n) is bounded in X. Arguing by contradiction we assume that, passing eventually to a subsequence, still denote by (u_n), $\|u_n\| \to \infty$ and $\|u_n\| > 1$ for all n. By (3.6) and (M_0), (M_1), for n large enough, we have

$$\begin{aligned}
1 + C_2 &\ge J_\lambda(u_n) - \frac{1}{r^-}\langle J'_\lambda(u_n), u_n\rangle + \frac{1}{r^-}\langle J'_\lambda(u_n), u_n\rangle \\
&\ge (1-d)M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\right)\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x - \lambda\int_\Omega \frac{a(x)}{q(x)}|u_n|^{q(x)}\,\mathrm{d}x \\
&\quad -\lambda\int_\Omega \frac{b(x)}{r(x)}|u_n|^{r(x)}\,\mathrm{d}x - \frac{1}{r^-}M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\right)\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x \\
&\quad +\frac{\lambda}{r^-}\int_\Omega a(x)|u_n|^{q(x)}\,\mathrm{d}x + \frac{\lambda}{r^-}\int_\Omega b(x)|u_n|^{r(x)}\,\mathrm{d}x + \frac{1}{r^-}\langle J'_\lambda(u_n), u_n\rangle \\
&\ge \frac{m_0}{\nu(p^+)^{\nu-1}}\left(\frac{1-d}{p^+} - \frac{1}{r^-}\right)\|u_n\|^{\nu p^-} - \lambda\left(\frac{1}{q^-} - \frac{1}{r^-}\right)C_3|a|_{\alpha(x)}\|u_n\|^{q^+} \\
&\quad -\frac{1}{r^-}\|J'_\lambda(u_n)\|_{X'}\|u_n\| \\
&\ge \frac{m_0}{\nu(p^+)^{\nu-1}}\left(\frac{1-d}{p^+} - \frac{1}{r^-}\right)\|u_n\|^{\nu p^-} - \lambda\left(\frac{1}{q^-} - \frac{1}{r^-}\right)C_3|a|_{\alpha(x)}\|u_n\|^{q^+} - \frac{C_4}{r^-}\|u_n\|.
\end{aligned}$$

Dividing the above inequality by $\|u_n\|^{\nu p^-}$, taking into account (2.1) holds true and passing to the limit as $n \to \infty$, we obtain a contradiction. It follows that (u_n) is bounded in X. By the reflexivity of X, for a subsequence still denoted (u_n), we have $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^{h(x)}(\Omega)$, where $1 \le h(x) < p^*(x)$. Therefore,

$$< J'_\lambda(u_n), u_n - u > \to 0, \tag{3.7}$$

$$\int_\Omega a(x)|u_n|^{q(x)-2}u_n(u_n - u)\,\mathrm{d}x \to 0, \tag{3.8}$$

and

$$\int_\Omega b(x)|u_n|^{r(x)-2}u_n(u_n - u)\,\mathrm{d}x \to 0. \tag{3.9}$$

Since (u_n) is bounded in X, passing to a subsequence, if necessary, we may assume that $\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x \to h_0 \ge 0$ as $n \to \infty$. If $h_0 = 0$ then (u_n) converges strongly to $u = 0$ in X and the proof is complete. If $h_0 > 0$ then since the function M is continuous, we obtain

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\right) \to M(h_0) \ge 0 \quad \text{as } n \to \infty.$$

Thus, by (M_0), for sufficiently large n, we have

$$0 < C_5 \le M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x\right) \le C_6. \tag{3.10}$$

From (3.7), (3.8), (3.9) and (3.10), we deduce that $A(u) := \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)\,\mathrm{d}x \to 0$. According to the fact that A satisfies Condition (S^+) (see [17]), we have $u_n \to u$ in X. This completes the proof. □

Proof of Theorem 3.1 From Lemmas 3.2 and 3.3, we deduce

$$\max(J_\lambda(0), J_\lambda(e)) = J_\lambda(0) < \inf_{\|u\|=\rho} J_\lambda(u) =: \beta.$$

By Lemma 3.4 and the Mountain Pass Theorem (see [1]), we deduce the existence of critical points u of J_λ associated of the critical value given by

$$c := \inf_{g\in\Gamma} \sup_{t\in[0,1]} J_\lambda(g(t)) \geq \beta, \tag{3.11}$$

where $\Gamma = \{g \in C([0,1], X) : g(0) = 0 \text{ and } g(1) = e\}$. This completes the proof. □

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